Formal languages and automata

Regular languages and

Regular Expression

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Regular language

- Suppose L is a language, as mentioned previously, L is a regular language if and only if we can design a DFA for it.

Example1

- $L = \{awa \mid w \in \{a, b\}^*\}$

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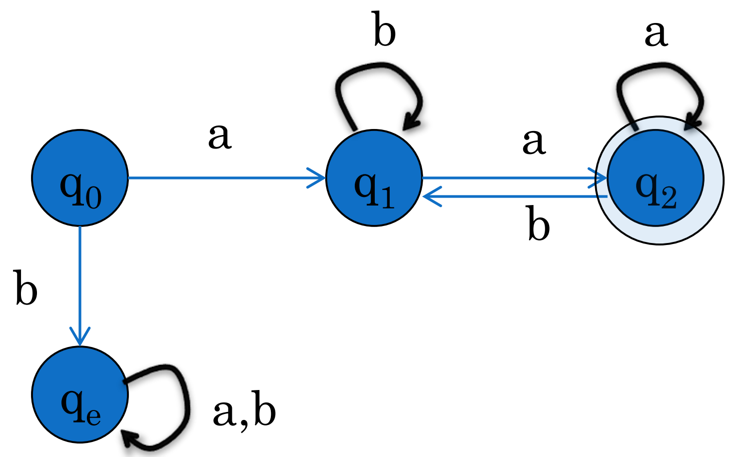

Example2

- $L^2 = \{awaawa \mid w \in \{a,b\}^*\}$

Example3

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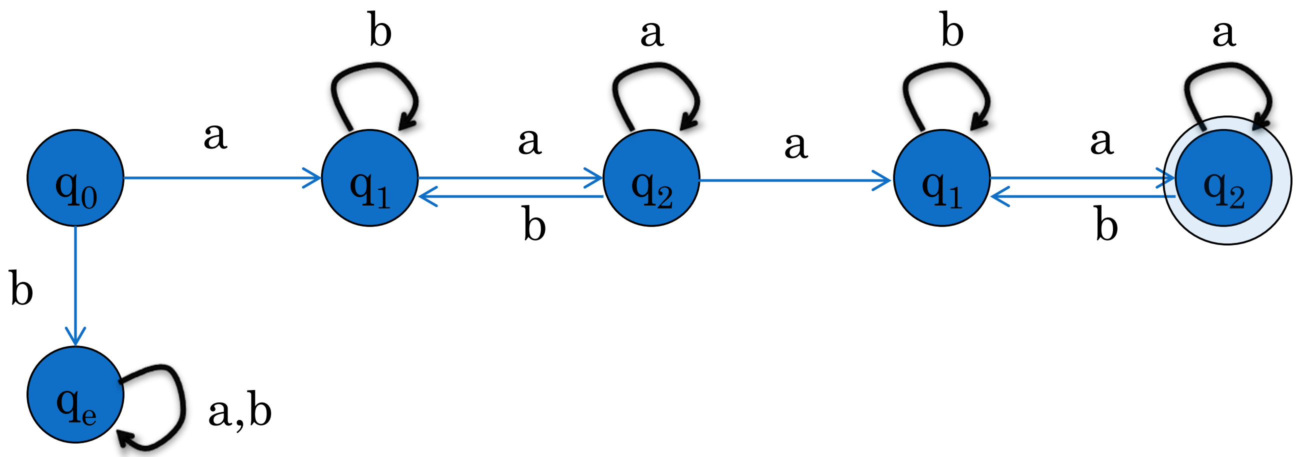

Note: if L is a regular language then $L^2, L^3, \ldots L^*$ are regular, too

Example4

- $L = \{a^n b^m | \, n \geq 2 \, , m \geq 1\}$

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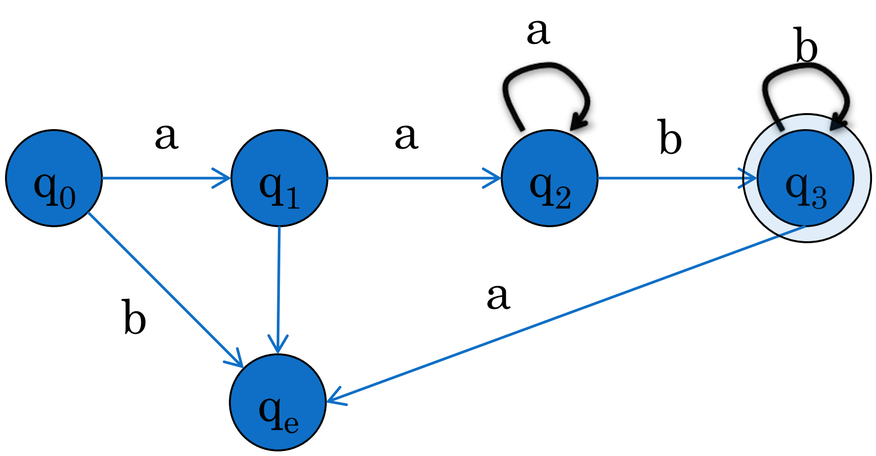

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- Thus, DFAs and Regular Expressions are equivalent.

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- $R^+ = RR^*$ and R^k for k-fold concatenation are useful shorthands.

RE examples

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RE examples

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RE examples

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$(a + b + c)$	⇦	$L = \{a, b, c\}$

RE examples

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- $= \{\varepsilon, a, bc, abc, bca, aa, bcbc, aaa, \ldots\}$

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$(\mathbf{0}^*\mathbf{1}\mathbf{0}^*\mathbf{1})^*\mathbf{0}^*$	$\rightarrow$	$\{\omega \mid n_1(\omega)$ is even$\}$

Writing RE

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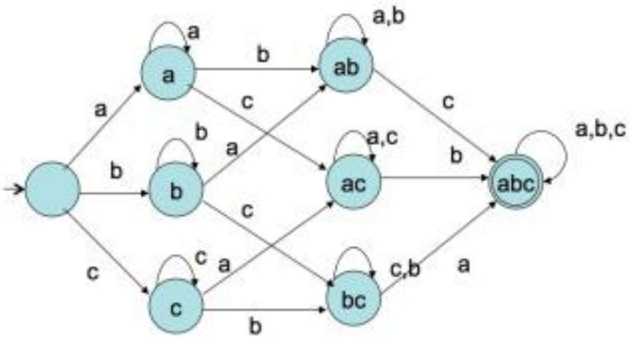

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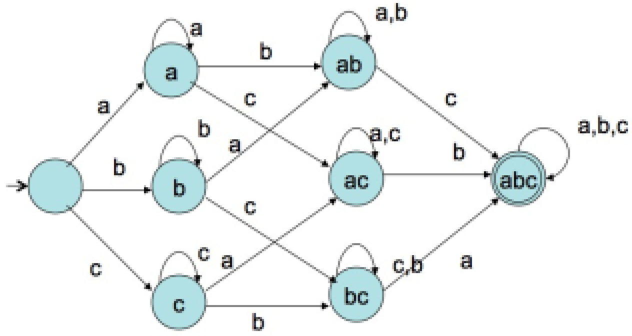

- DFAs and REs may need different ways of looking at the problem.
  - For the DFA, you count symbols
  - For the RE, you enumerate all possible patterns

RE identities

- $\mathbf{R} \cup \phi = \mathbf{R}$
- $\mathbf{R}\epsilon = \epsilon\mathbf{R} = \mathbf{R}$
- $\phi^* = \epsilon$
- Note that we do not have explicit operators for intersection or complementation!

Equivalence with FA

THEOREM

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Proof Idea

Inductively convert a given regular expression to an NFA.

Converting RE to NFA

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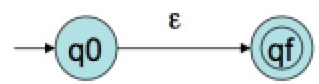
ϵ	

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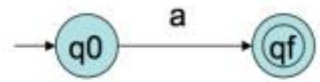
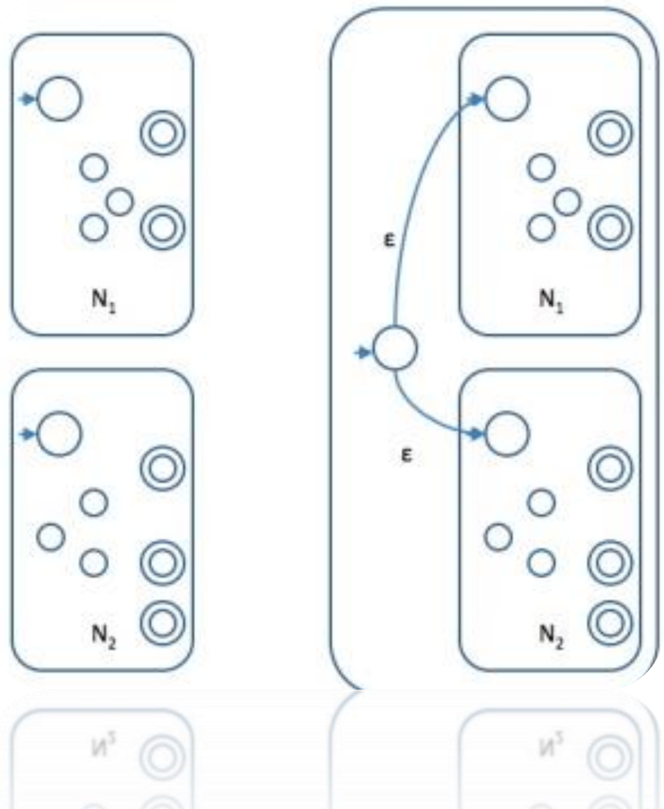
a for $a \in \Sigma$	

Union

- Let N_1 and N_2 be NFAs for R_1 and R_2 respectively. Then the NFA for $\mathbf{R_1} \cup \mathbf{R_2}$ is



Concatenation

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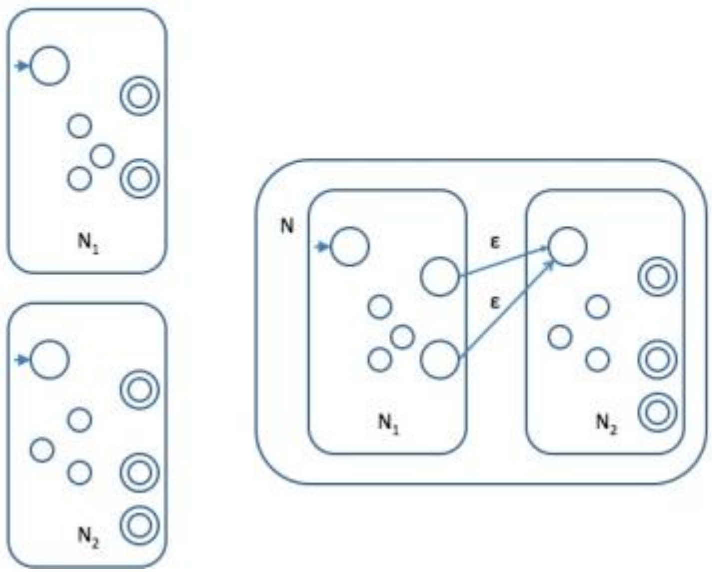
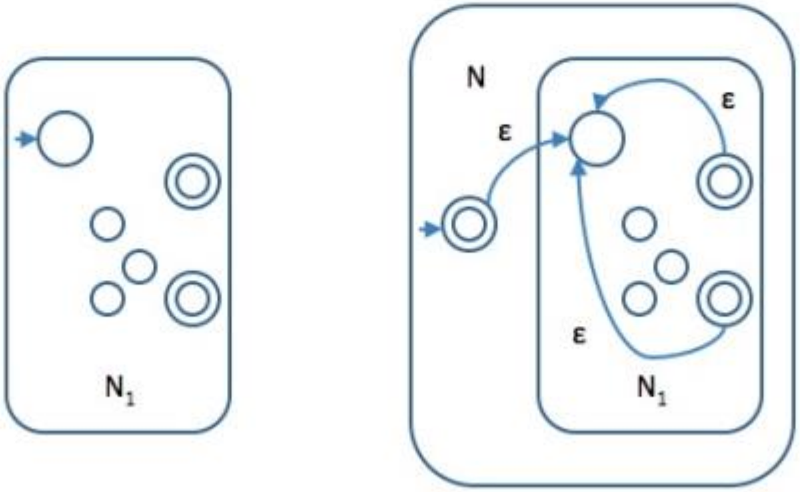


Star

- Let N be NFAs for R. Then the NFA for $\mathbf{R}^*$ is



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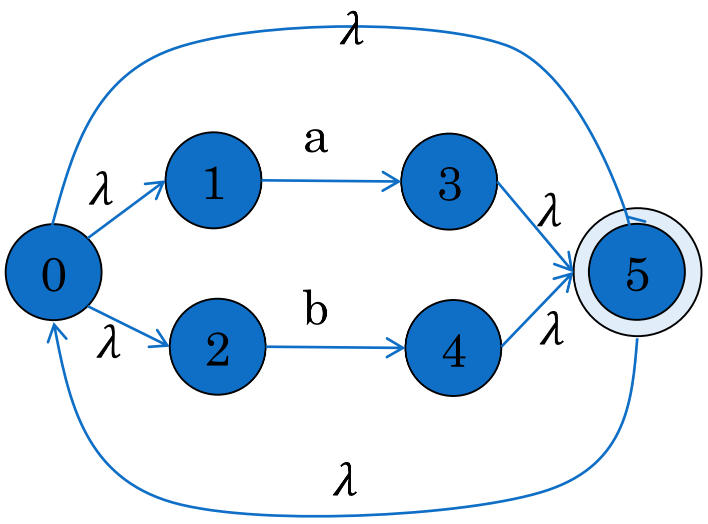

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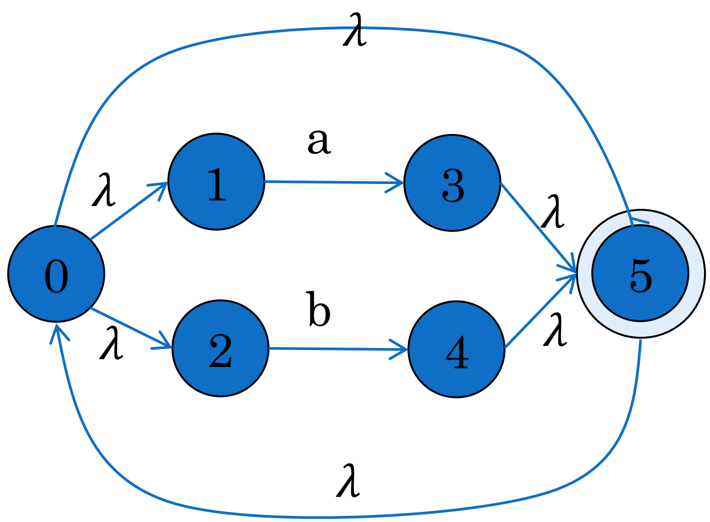
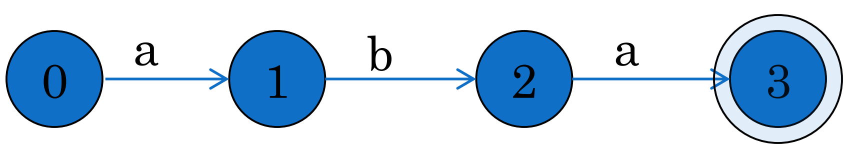
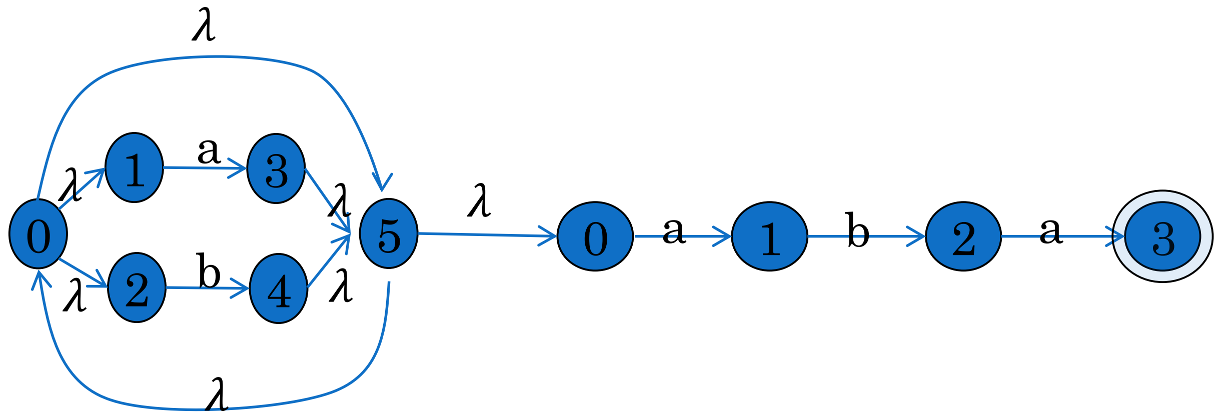




- Let's convert (**a** ∪ **b**)***aba** to an NFA.



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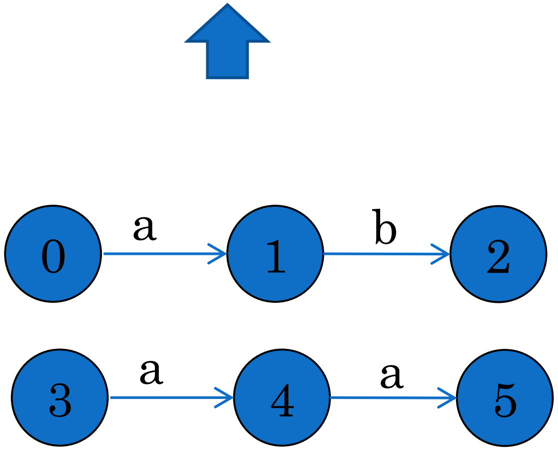

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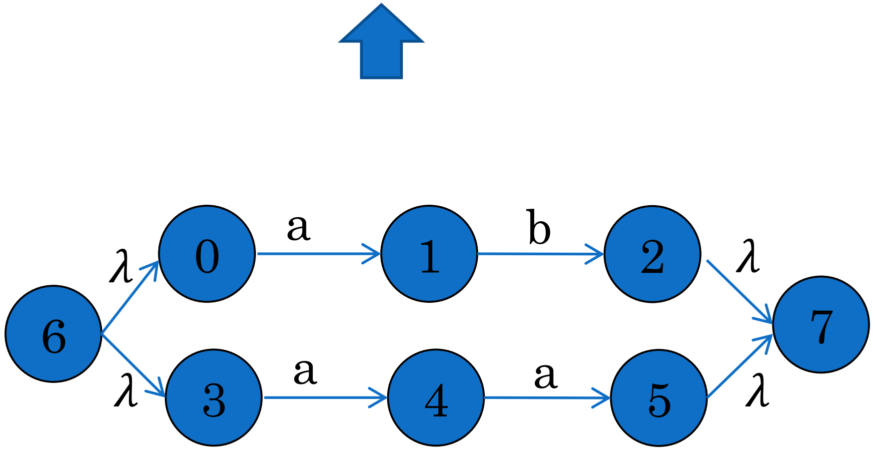

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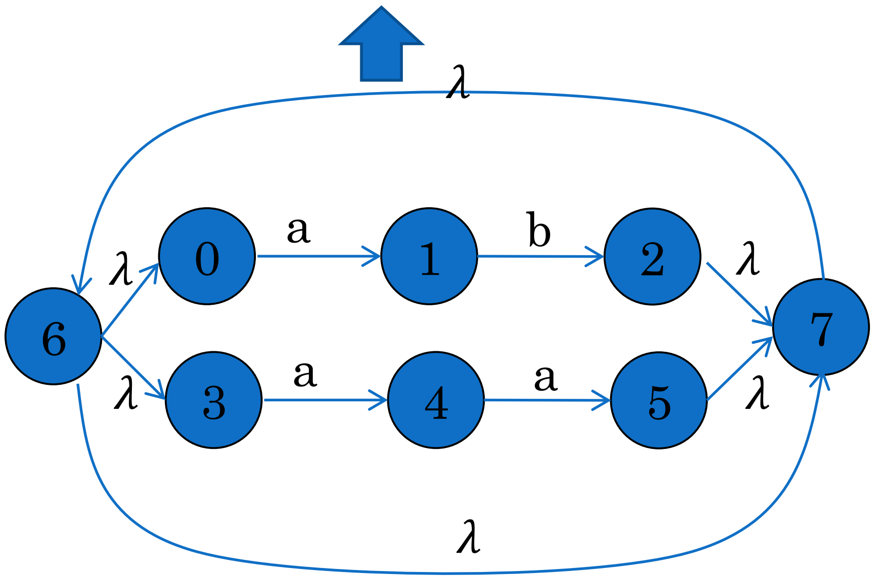

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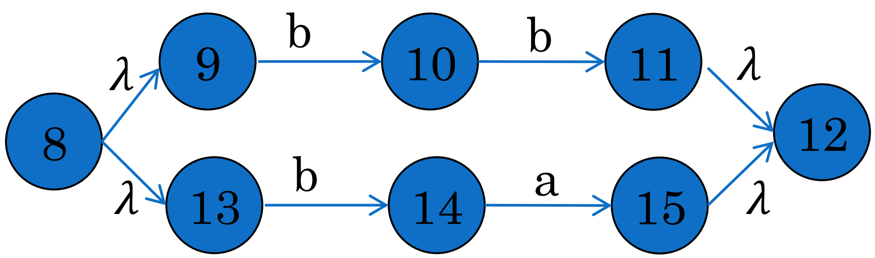

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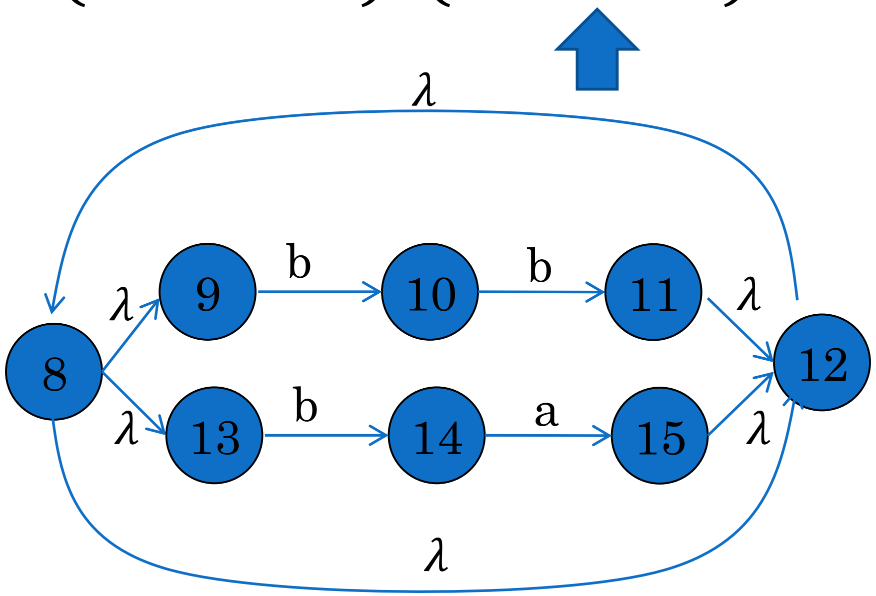

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